

Modeling of thin isotropic polygonal plates using macroelement method

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Abstract The article presents an analytical-numerical macroelement method for modeling and solving thin, isotropic plates with a convex polygon shape and various boundary conditions. The developed method consists of mathematical, computational and geometric models. It was implemented in a computer program to solve the boundary problem of plate structures under continuous load. Results for triangular, skewed, trapezoidal plates are presented as examples of solutions. Based on their analysis, the advantages and limitations of the method are presented. Directions for further development of the method were also indicated.

INTRODUCTION

Bending problems of thin plates are usually solved using Classical (Kirchhoff) Plate Theory. This theory in many cases correctly describes the behavior of plate structures, from thin to moderately thick plates, and gives well-founded results that are sufficient for practice. Methods for solving plate structures are divided into two basic groups: analytical and numerical. These methods were analyzed in the monographs [1, 2, 3].

Analytical methods [2, 4, 5, 6, 7, 8, 9] are based on the continuum concept, while numerical methods [2, 10, 11, 12, 13, 14] are based on discrete models. Analytical methods are the most rigorous ones, providing exact solutions, but are limited to plates with simple shapes. Numerical methods allow solving structures of any configuration for which analytical solutions are complex or impossible. The basis of numerical methods is the variational approach [9]. Although they give approximate solutions, they are accurate enough for engineering purposes. Their accuracy depends on the number of elements, the type of elements, the adopted assumptions as well as the experience of the software user. Numerical methods have inherent errors (e.g., the geometry of the structure is approximate, the computer carries only a finite number of digits, the combination of elements with very large stiffness differences yields numerical difficulties) [15, 16]. Increasing the number of solution parameters leads to an increase in computation time and the accumulation of computational errors. Numerical methods include: Finite Element Method (FEM) [13, 17, 18], Boundary Element Method (BEM) [19, 20, 21], Trefftz collocation method [22, 23, 24, 25], least square collocation method [26] and others. There is also increasing interest in meshless methods [27, 28, 29, 30, 31]. In MMs the spatial domain is represented by a set of nodes (cloud of points) and not discretized by elements as in most of the mesh-based methods (finite difference method, finite element method, finite volume method); consequently, there is no need for predefined connectivity between the nodes [32].

In recent years, many methods have been proposed that are free from the disadvantages of standard numerical methods. According to these methods, some of the equations are performed analytically, and some using numerical procedures. The proposed method for solving thin, isotropic polygonal plates belongs to this group. It combines the advantages of analytical methods (Navier and Lévy) and numerical methods (including meshless methods).

The basis of this method is the plate macroelement defined on the authors' publications [33, 34, 35, 36, 37, 38, 39]. To solve the boundary problem of thin polygonal plates, a non-symmetric model was used. This universal model can be used for symmetrical [37] and non-symmetrical plate configurations. The symmetry of the plate configuration refers to the shape of the plate, elastic properties, outer load and boundary conditions. The previous articles analyzed rectangular plates with various boundary conditions [39] and some special cases of non-rectangular plates. The aim of the work is to develop an universal and easy-to-use method for solving plate structures of any shape, but we will limit our considerations to convex polygons.

MATHEMATICAL MODEL OF A PLATE

The mathematical model is based on an exact solution of the plate equilibrium equation within the continuum concept. The kinematic and static boundary conditions are satisfied at separate nodes at the edges of the plate. Therefore, the plate is modeled as a continuous medium in the inner region and as a discrete object at the edges.

Equilibrium state of thin isotropic plate is reduced to solution of non-uniform fourth-order partial differential equation [4]

$$\nabla^2 \nabla^2 w = \frac{q}{D}, \quad (1)$$

where function $w(x_1, x_2)$ is the deflection of the plate and D is its flexural rigidity. Function $q(x_1, x_2)$ describes distribution of the outer active load on the upper surface of the plate while its bottom surface is unloaded. Considering a variables separation technique, we select the solution of equation (1) as the sum of general solution w_0 of uniform equation $\nabla^2 \nabla^2 w = 0$ and some particular solution w_* of equation (1).

$$w = w_0 + w_* \quad (2)$$

General solution w_0 is taken in the form [37]

$$w_0 = f_{kps}(x_s) \cdot T_{kp(3-s)}(x_{3-s}), \quad (3)$$

where $f_{kps}(x_s)$ are unknown functions and $T_{kps}(x_s)$ are given trigonometric functions:

$$T_{kps} = \begin{cases} \cos(\kappa_{kps} x_s), & p = 1, 2 \\ \sin(\kappa_{kps} x_s), & p = 3, 4 \end{cases} \quad (4)$$

In the above $k = 1, \dots, K$; $p = 1, \dots, 4$, $s = 1, 2$, where K is parameter which determines accuracy of the solution. The K is greater the accuracy is better. Einstein summation rule is used here. The κ_{kps} parameters are as follows:

$$\kappa_{kps} = \begin{cases} \gamma_{ks} = k\pi/a_s, & p = 1, 3 \\ \delta_{ks} = (2k-1)\pi/2a_s, & p = 2, 4 \end{cases} \quad (5)$$

where a_s are the sizes of the plate macroelement.

Functions $f_{kps}(x_s)$ included in formula (3) are taken in the form:

$$f_{kps}(x_s) = R_{kps} \cdot E_{kps}(x_s). \quad (6)$$

Here R_{kps} are unknown coefficients. Because indices k, p, s appear at the both sides of this formula, summation according to these indices does not perform. We expect the functions $E_{kps}(x_s)$ as:

$$E_{kps}(x_s) = \exp(\lambda_{kps} x_s), \quad (7)$$

and come to linear algebraic equation with respect to unknown parameters λ_{kps} .

$$\lambda_{kps}^4 - 2\kappa_{kp(3-s)}^2 \lambda_{kps}^2 + \kappa_{kp(3-s)}^4 = 0, \quad (8)$$

Let λ_{kpsv} , $v = 1, \dots, 4$ be roots of equations (8). Finally functions $f_{kps}(x_s)$ take the form [37]

$$f_{kps}(x_s) = R_{kpsv} \cdot B_{kpsv}(x_s), \quad (9)$$

where

$$B_{kpsv} = \begin{cases} \cosh(\kappa_{kp(3-s)} x_s), & v = 1 \\ \frac{x_s}{a_s} \sinh(\kappa_{kp(3-s)} x_s), & v = 2 \\ \sinh(\kappa_{kp(3-s)} x_s), & v = 3 \\ \frac{x_s}{a_s} \cosh(\kappa_{kp(3-s)} x_s), & v = 4 \end{cases} \quad (10)$$

are basic functions of the solution.

According to (9) we present general solution of equation (1) in the form:

$$w_0(x_1, x_2) = R_{kpsv} \cdot W_{kpsv}(x_1, x_2). \quad (11)$$

Introduced functions

$$W_{kpsv}(x_1, x_2) = B_{kpsv}(x_s) \cdot T_{kp(3-s)}(x_{3-s}) \quad (12)$$

are called shape functions of the plate deflection. Unknown coefficients R_{kpsv} are degrees of freedom (DOF) of the plate deflection.

Particular solution of equation (1) describes conditions of equilibrium between internal reactions and outer load applied at each surface nodes of the plate. Because mathematical model deals only with continuous functions, outer load $q(x_1, x_2)$ in the right side of equation (1) must be continuous and be a function of class C^k , where k is the order of governing equation (the first k derivatives all exist and are continuous). In fact, it can be described in continuous or discrete form. For this reason, we replace it with a continuous load function $Q(x_1, x_2)$ presented in the form of tensor polynomial:

$$Q(x_1, x_2) = A_0 + A_{mps} T_{mps}(x_s) + B_{mnpq} T_{mnpq}(x_1, x_2). \quad (13)$$

Functions $T_{mps}(x_s)$ and $T_{mnpq}(x_1, x_2) = T_{mp1}(x_1) \cdot T_{nq2}(x_2)$ are single and double trigonometric functions, where $m = 1, \dots, M$, $n = 1, \dots, N$, $p = 1, \dots, 4$, $q = 1, \dots, 4$, $s = 1, 2$. The functions $T_{mp1}(x_1)$ and $T_{nq2}(x_2)$ are obtained using the relationship (4). Unknown coefficients A_0 , A_{mps} , B_{mnpq} are determined from conditions of equality of loading function $Q(x_1, x_2)$ and outer load $q(x_1, x_2)$ written at the separate points (x_1^*, x_2^*) of the plate surface.

$$Q(x_1, x_2) \Big|_{\substack{x_1=x_1^* \\ x_2=x_2^*}} = q(x_1^*, x_2^*). \quad (14)$$

We choose particular solution of equation (1) in a similar form

$$w_*(x_1, x_2) = A_0^* \Psi(x_1, x_2) + A_{mps}^* T_{mps}(x_s) + B_{mnpq}^* T_{mnpq}(x_1, x_2). \quad (15)$$

where $\Psi(x_1, x_2)$ is an algebraic polynomial of forth order:

$$\Psi(x_1, x_2) = \frac{x_1^4}{24a_1^4} + \frac{x_1^2 x_2^2}{4a_1^2 a_2^2} + \frac{x_2^4}{24a_2^4}. \quad (16)$$

Functions included in this formula are called loading functions of the plate deflection.

The derivation of the formulas given above is presented in the dissertation [38]. In this way, we obtain the function for plate deflection (2) from formulas (11) and (15). The developed model provides a compact analytic solution capable of easy numerical evaluation.

COMPUTATIONAL MODEL OF A PLATE

The mathematical model described in the previous section was implemented in the form of a computer program written in the Python programming language. The proposed implementation closely follows the mathematical model. Partial derivatives are calculated using Automatic Differentiation [40, 41, 42]. The program's modular structure and concise code make it easier to understand how it works, test individual parts and make modifications.

The program takes some techniques from functional programming [43], such as closures. Programming in a functional style often includes using functions as values by passing them in arguments, returning them from other functions, assigning them to variables for later execution, and so forth.

Using relations known from theory of thin isotropic plates [44] and automatic differentiation, we get functions for slopes $\phi_1(x_1, x_2)$, $\phi_2(x_1, x_2)$, bending and twisting moments $M_{11}(x_1, x_2)$, $M_{22}(x_1, x_2)$, $M_{12}(x_1, x_2)$ shearing forces $Q_1(x_1, x_2)$, $Q_2(x_1, x_2)$ and generalized shearing forces $V_1(x_1, x_2)$, $V_2(x_1, x_2)$. They are called functions of state of the plate and constitute a computational model of the plate. These functions can be written in the general form:

$$F(x_1, x_2) = R_{kpsv} F_{kpsv}(x_1, x_2) + F_*(x_1, x_2), \quad (17)$$

where $F(x_1, x_2)$ denotes one of the following quantities:

$$F = (w, \varphi_1, \varphi_2, \varphi_n, M_{11}, M_{22}, M_{12}, M_n, Q_1, Q_2, V_1, V_2). \quad (18)$$

These quantities are mapped to the functions included in the model

$$\begin{aligned} w &\rightarrow W, & \varphi_1 &\rightarrow U, & \varphi_2 &\rightarrow V, & \varphi_n &\rightarrow P, \\ M_{11} &\rightarrow X, & M_{22} &\rightarrow Y, & M_{12} &\rightarrow Z, & M_n &\rightarrow F, \\ Q_1 &\rightarrow G, & Q_2 &\rightarrow H, & V_1 &\rightarrow K, & V_2 &\rightarrow L, \end{aligned} \quad (19)$$

where $U = U_{kpsv}(x_1, x_2)$, $V = V_{kpsv}(x_1, x_2)$ and so on are the shape functions while quantities with * are force functions of slopes, moments and shearing forces.

Expression (17) contains two sets of arbitrary coefficients that allow to satisfy boundary and surface conditions with a given accuracy.

Since expression (13) is exact solution of equilibrium equation (1), internal forces and displacements within the plate are balanced with outer load. We satisfy these conditions at the plate surface at points called surface nodes. Hence, we determine particular solution of equation (1). But edge quantities are imbalanced because expression (11) contains degrees of freedom of the plate R_{kpsv} . In order to stabilize the plate some constraints in the form of boundary conditions should be imposed at separate points at the plate contour. We call these points edge nodes. Each constraint corresponds to one parameter R_{kpsv} .

Finally we reduce solution of the problem to system of algebraic equations with respect to unknown coefficients R_{kpsv} . Solving the system of equations, we eliminate degrees of freedom of the plate. Then we calculate required static and kinematic quantities and print results, usually in the form of plots (2D or 3D).

The source code of the program implemented described computational model used to obtain the results presented in this article is available in the repository: https://github.com/krysros/mcr_2024.

GEOMETRIC MODEL OF A PLATE

The geometry of the plate is defined by the coordinates of the subsequent vertices V_i of the polygon coinciding with the contour of the plate. The contour of the plate is limited by a rectangle, which may be an envelope or the minimum rotated rectangle that encloses an input geometry, such that the resulting rectangle has minimum area.

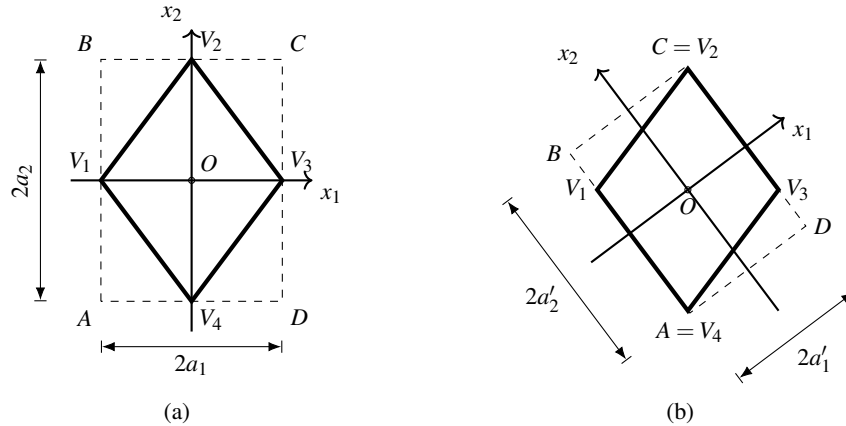


Figure 1: The bounding rectangle of the plate macroelement: (a) the envelope and (b) the minimum rotated rectangle.

In some cases, the choice of bounding rectangle determines whether the problem is symmetrical or non-symmetrical. By introducing a rectangular contour we narrow the domain of the problem to $D = [-a_1, a_1] \times [-a_2, a_2]$ as shown in Figure 1.

In order to obtain results using the computational model, boundary and surface nodes must be selected. The total number of DOF for parameter K is $n = 32K$, so we have to put $32K$ constraints at the plate contour in the form of

boundary conditions. Because two boundary conditions are written at each edge node, thus $16K$ edge nodes must be generated at the plate contour. The total number of surface points must be equal to number of coefficients included in equation (15)

$$j = 16MN + 4M + 4N + 1. \quad (20)$$

The number of nodes is predetermined. The problem is finding the optimal way to arrange them. We usually aim for an even distribution of nodes, but this is only possible in certain cases.

Edge nodes

In this article, we propose to determine the number of nodes for each edge and then distribute them evenly. For this purpose, we select $4K$ initial points on the axes of the coordinate system, in intervals bounded by a rectangular contour. We then project these points onto the edges of the plate.

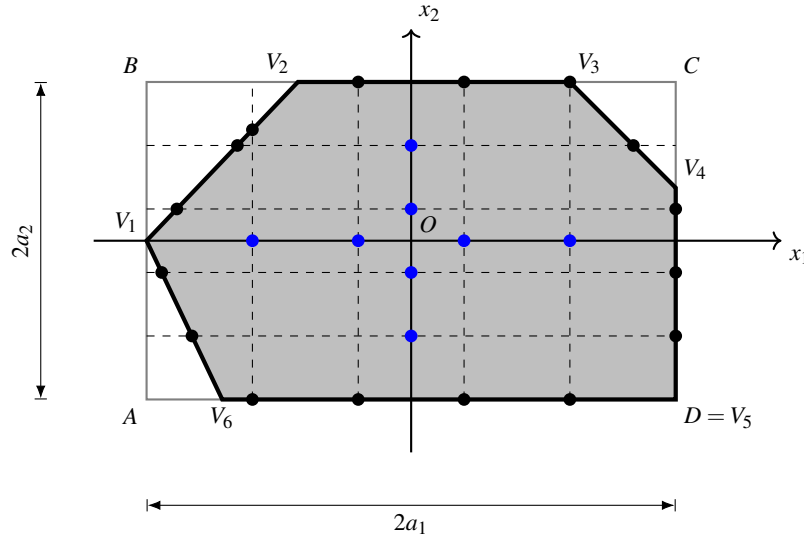


Figure 2: Determining the number of edge nodes by projecting initial points.

This method for $K = 1$ is presented graphically in Figure 2. We do not include the start and end point. In practice, we divide the segment into $4K + 2$ points and discard the first and last of them. The coordinates of the projections of these points are found as the intersection points of the edges and lines passing through these points and parallel to the axis of the coordinate system. If the projection of a point hits the vertex of the plate between edges j and $j + 1$, we assume that it creates a node only on the j -th edge. This situation occurs in the V_3 vertex of the plate in Figure 2. However, if the projections of the initial points from the x_1 and x_2 axes coincide at the same point, we assume that they form two separate nodes. The number of points on the edges obtained in this way is presented in Table 1.

Table 1: The table shows the number of nodes on the edges of plate from Figure 2

No.	Edge	Number of nodes
1	V_1-V_2	3
2	V_2-V_3	3
3	V_3-V_4	1
4	V_4-V_5	3
5	V_5-V_6	4
6	V_6-V_1	2

Then we distribute a certain number of nodes evenly on the edges, as shown in Figure 3.

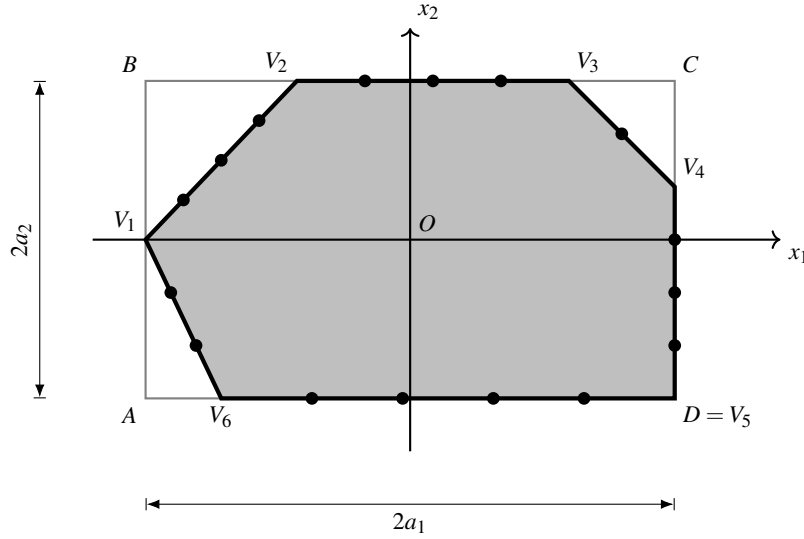


Figure 3: Uniform distribution of a given number of nodes on the edges.

Surface nodes

For plates with simple, regular shapes, it is possible to find a way to evenly distribute surface nodes. In any case, we can use a random selection of points in the area bounded by the rectangular contour. Examples of such distributions are shown in Figure 4.

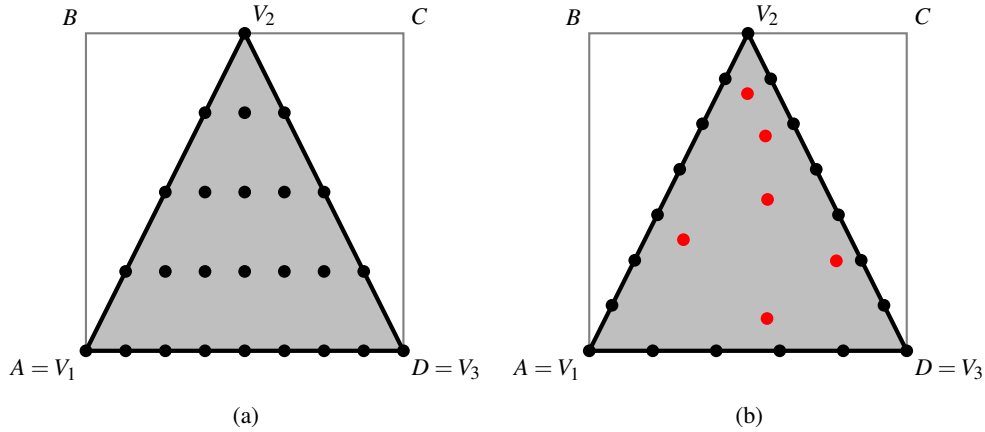


Figure 4: Uniform (a) and random (b) distribution of surface nodes of a triangular plate for $K = 1$ and $M = N = 1$.

Since the coordinates of the plate vertices and edge nodes are known, we can use them for surface nodes. We generate the coordinates of the missing number of nodes randomly. If a random point is located inside the polygon, it is included in the set of surface nodes. Otherwise, the next point is randomly selected.

Selection of surface nodes does not significantly affect the accuracy of the solution. To improve the results, it is proposed to use the point alignment method, e.g. the Lloyd's algorithm, also known as Voronoi iteration or relaxation, for finding evenly spaced sets of points in subsets of Euclidean spaces and partitions of these subsets into well-shaped and uniformly sized convex cells [45].

RESULTS

The article presents the selected results obtained for plates from Figure 5. The considered plates are isotropic, simply supported on the contour and loaded with a uniformly distributed load on their upper surface. The following geometric and mechanical model parameters were used for calculations: thickness $h = 0.2$ m, Young's modulus $E = 30 \times 10^9$ Pa and Poisson ratio $\nu = 0.2$. The intensity of the load q_0 is equal to 10 kPa.

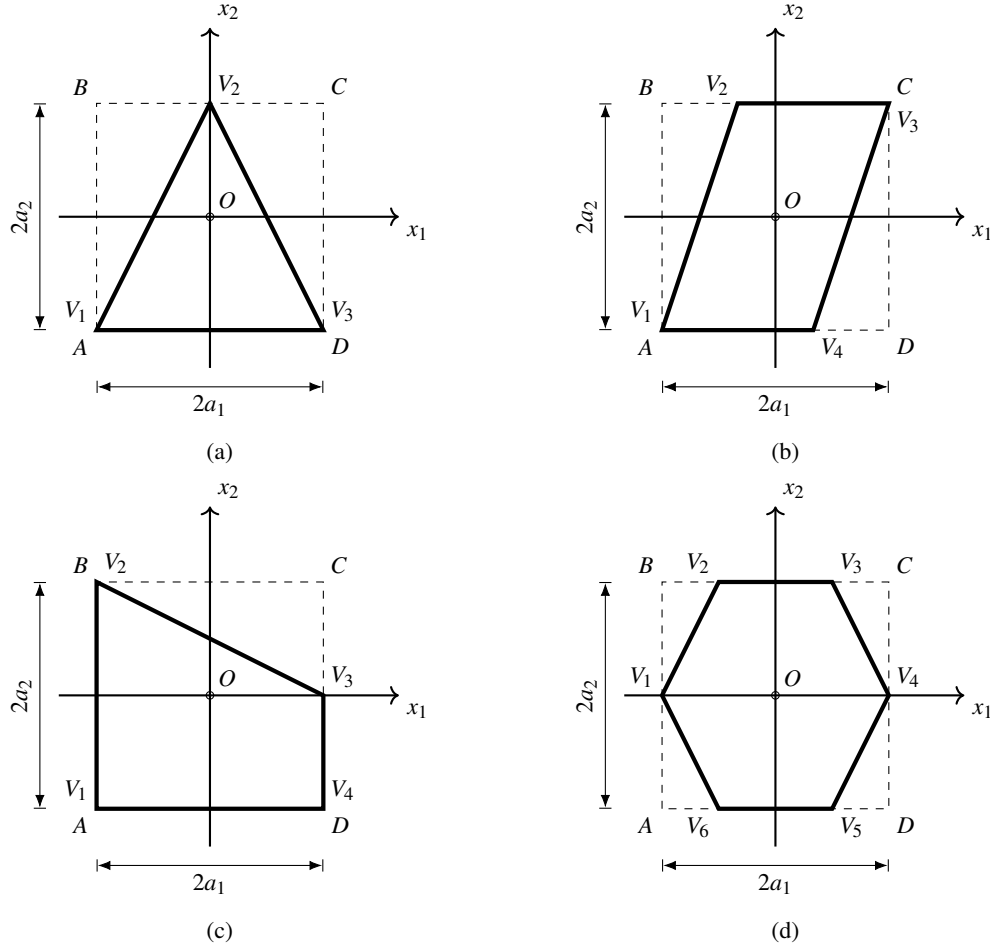


Figure 5: Examples of plate macroelements: (a) triangular, (b) skewed, (c) trapezoidal, and (d) hexagonal

Triangular plate

Let us consider a simply supported triangular plate (Figure 5a) with vertex coordinates $V_1 = (-3, -3)$, $V_2 = (0, 3)$, $V_3 = (3, -3)$. For this example, we will present and compare the results for a uniform and random distribution of surface nodes ($M = N = 5$).

Plots 6 show the deflection of a triangular plate for uniform and random distribution of surface nodes. The minimum and maximum values of kinematic and static quantities for triangular plate with uniform and random distribution of nodes are presented in Table 2. The differences between these values do not exceed 1 %.

The present method also produces results for the rotation angle φ_n and bending moments M_n , which depend on the angle α between the normal $\vec{n}$ and the x -axis. This allows you to verify the accuracy of meeting the boundary conditions on skew edges.

Let's assume that the angles α between the x -axis and the normals to the edges V_1 - V_2 and V_2 - V_3 are 153° and 27° , respectively. The values of the moments on the edges V_1 - V_2 (Figure 7a) and V_2 - V_3 (Figure 7b) are zero.

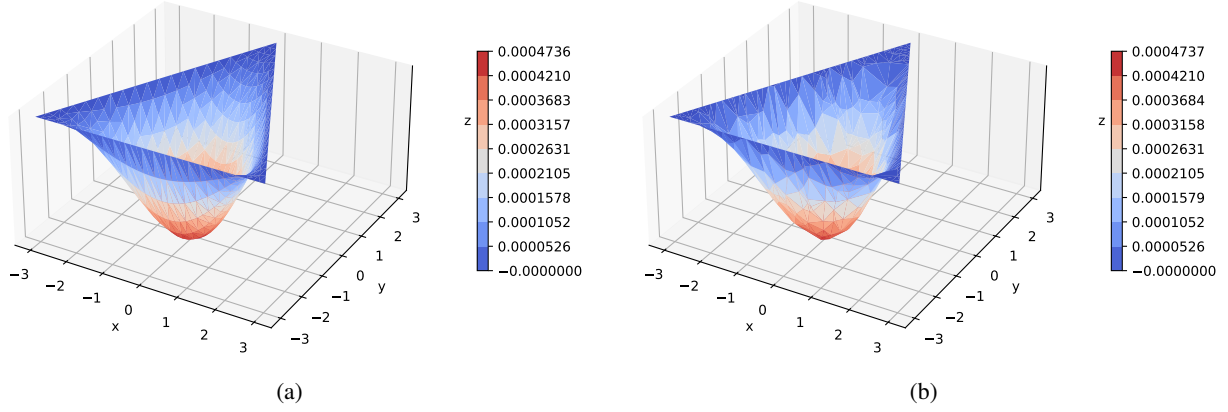


Figure 6: Three-dimensional plot of the deflection of a triangular plate with (a) uniform and (b) random distribution of surface nodes

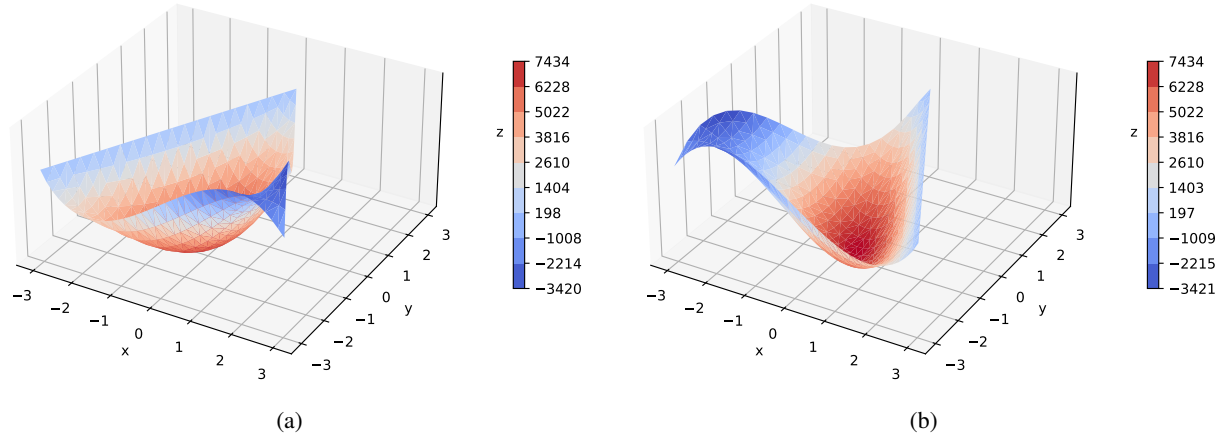


Figure 7: Three-dimensional plot of bending moment M_n of a triangular plate (a) $\alpha = 153^\circ$, (b) $\alpha = 27^\circ$

Table 2: Minimum and maximum values of kinematic and static quantities along with their relative error (RE) for a simply supported triangular plate with uniform and random distribution of nodes

Func.	Unit	Uniform		Random		RE	
		Min	Max	Min	Max	Min	Max
w	m	0	0.0004736	0	0.0004737	0 %	0.01 %
φ_1	rad	-0.000389	0.000389	-0.000390	0.000390	0.08 %	0.08 %
φ_2	rad	-0.000206	0.000424	-0.000208	0.000423	0.57 %	0.22 %
M_{11}	Nm/m	-2850.40	8006.10	-2872.85	7973.67	0.79 %	0.41 %
M_{22}	Nm/m	-2535.65	7019.75	-2538.61	7001.34	0.12 %	0.26 %
M_{12}	Nm/m	-3583.30	3584.22	-3579.05	3579.76	0.12 %	0.12 %
Q_1	Nm ⁻¹	-12499.10	12495.14	-12508.10	12504.25	0.07 %	0.07 %
Q_2	Nm ⁻¹	-6252.21	13803.93	-6257.24	13789.06	0.08 %	0.11 %
V_1	Nm ⁻¹	-14982.12	14976.67	-14989.79	14983.69	0.05 %	0.05 %
V_2	Nm ⁻¹	-9220.07	16746.01	-9220.07	16722.39	0 %	0.14 %

The plots of the remaining quantities listed in Table 2 are presented in Figure 8.

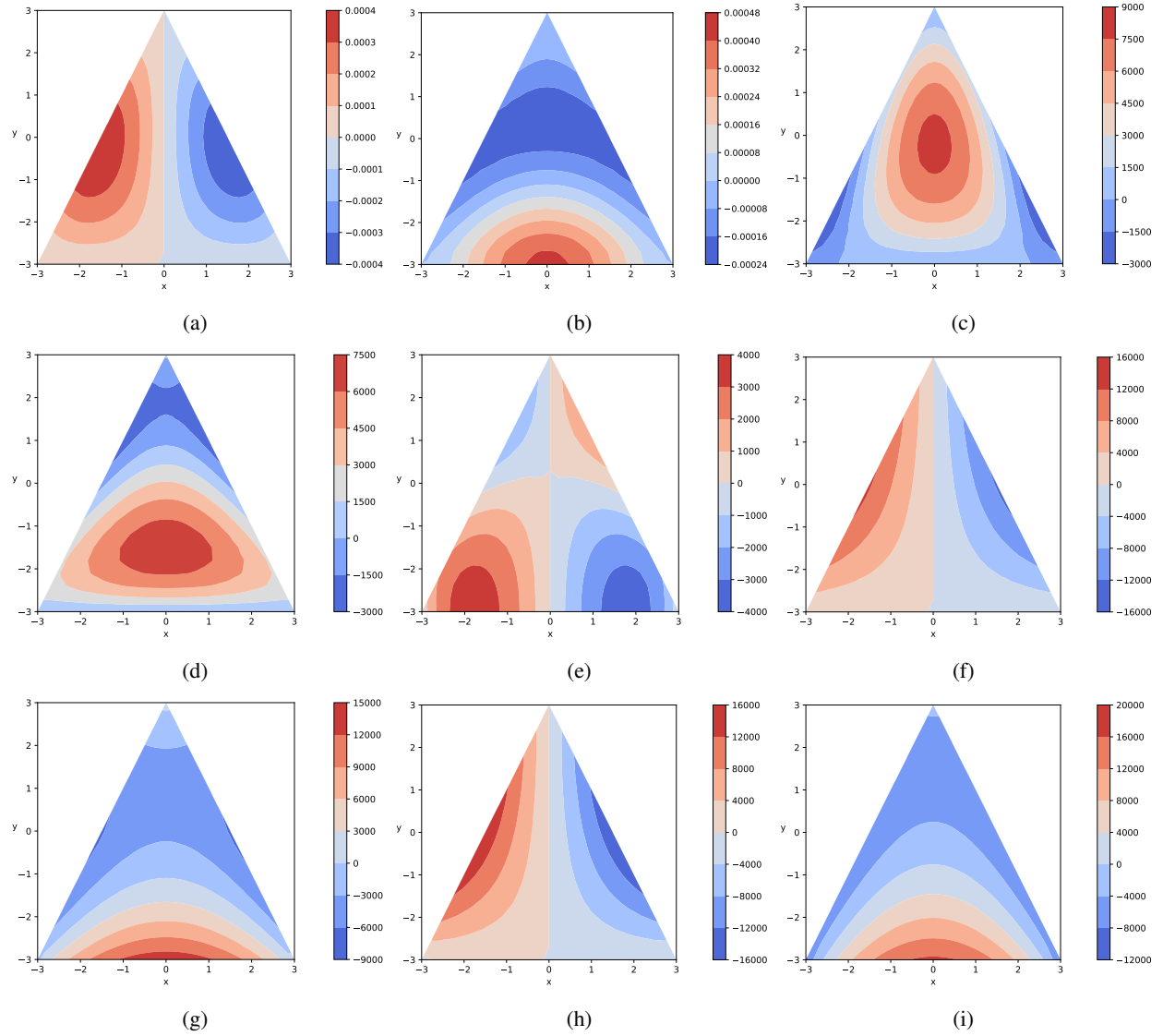


Figure 8: Functions for slopes (a) ϕ_1 , (b) ϕ_2 , bending and twisting moments (c) M_{11} , (d) M_{22} , (e) M_{12} , shearing forces (f) Q_1 , (g) Q_2 and generalized shearing forces (h) V_1 , (i) V_2 of a triangular plate

Four-sided plates

Parallelogram (Figure 5b) and trapezoidal (Figure 5c) plates were considered as examples of four-sided non-rectangular simply supported plates. Asymmetric variants were selected for analysis. In both cases, the polygon corresponding to the contour contains an obtuse angle.

A parallelogram plate is defined by the coordinates of its vertices

$$V_1 = (-3, -3), \quad V_2 = (-1.5, 3), \quad V_3 = (3, 3), \quad V_4 = (1.5, -3)$$

and the trapezoidal plate by the coordinates

$$V_1 = (-3, -3), \quad V_2 = (-3, 3), \quad V_3 = (3, 0), \quad V_4 = (3, -3).$$

Plots 9 and 11 show the deflection of parallelogram and trapezoidal plate for random distribution of surface nodes. The plots of the remaining quantities are presented in Figure 10 and 12.

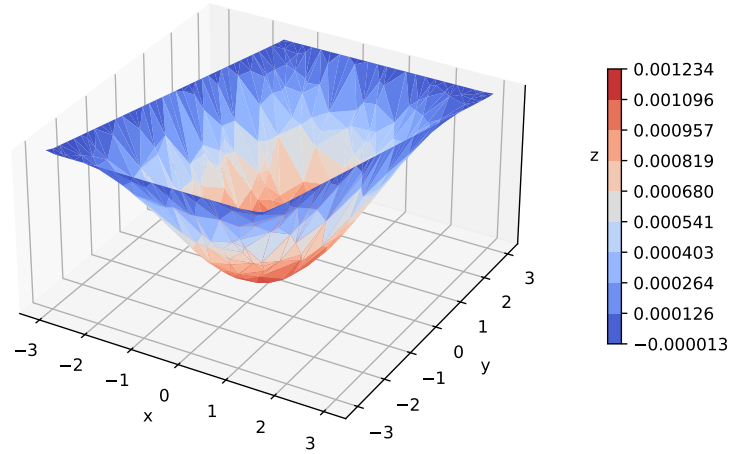


Figure 9: Three-dimensional plot of the deflection of a parallelogram plate with random distribution of surface nodes

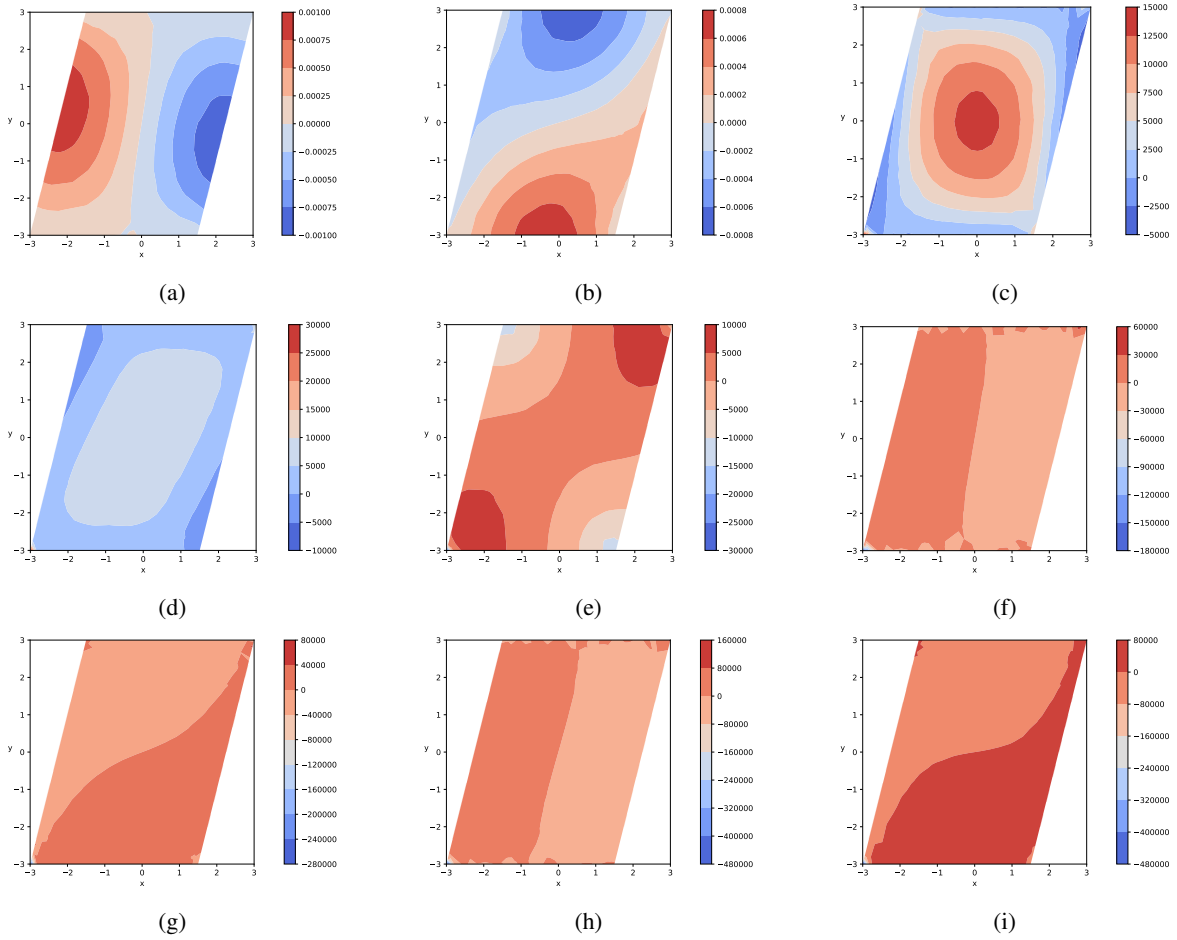


Figure 10: Functions for slopes (a) ϕ_1 , (b) ϕ_2 , bending and twisting moments (c) M_{11} , (d) M_{22} , (e) M_{12} , shearing forces (f) Q_1 , (g) Q_2 and generalized shearing forces (h) V_1 , (i) V_2 of a parallelogram plate

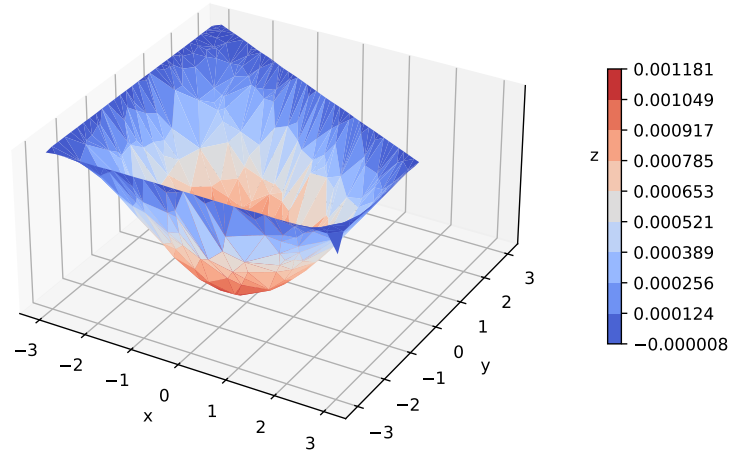


Figure 11: Three-dimensional plot of the deflection of a trapezoidal plate with random distribution of surface nodes

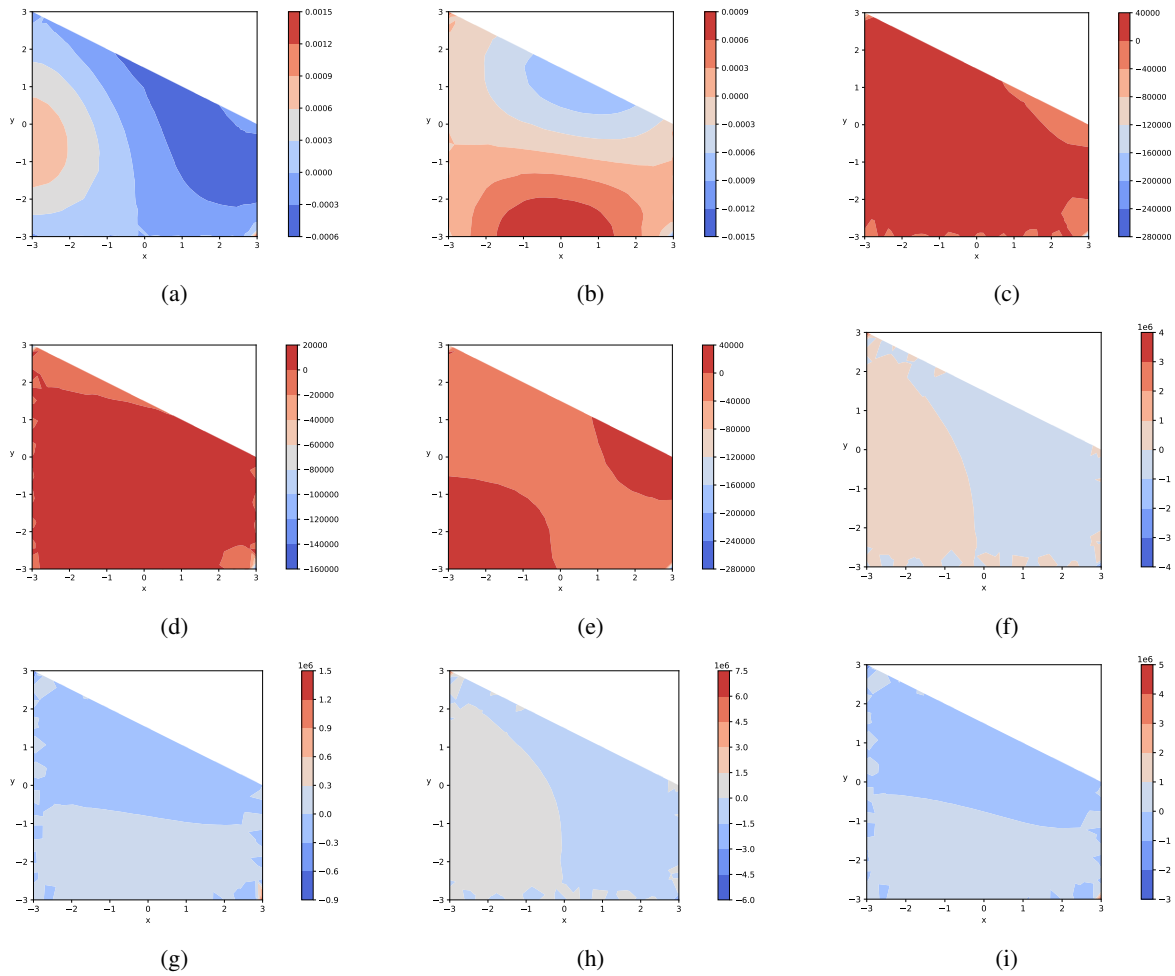


Figure 12: Functions for slopes (a) ϕ_1 , (b) ϕ_2 , bending and twisting moments (c) M_{11} , (d) M_{22} , (e) M_{12} , shearing forces (f) Q_1 , (g) Q_2 and generalized shearing forces (h) V_1 , (i) V_2 of a trapezoidal plate

Hexagonal plate

As a final example, let's consider a simply supported hexagonal plate (Figure 5d) with vertex coordinates $V_1 = (-3, 0)$, $V_2 = (-1.5, 2.598)$, $V_3 = (1.5, 2.598)$, $V_4 = (3, 0)$, $V_5 = (1.5, -2.598)$, $V_6 = (-1.5, -2.598)$.

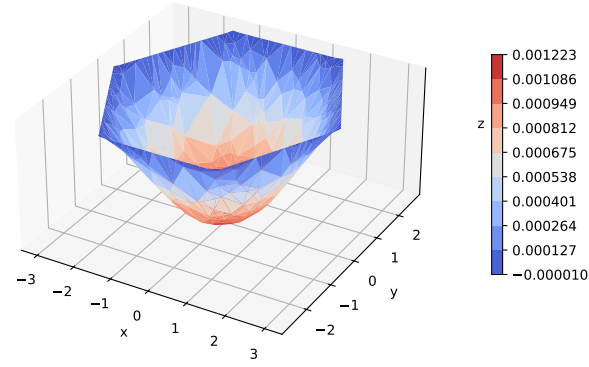


Figure 13: Three-dimensional plot of the deflection of a hexagonal plate with random distribution of surface nodes

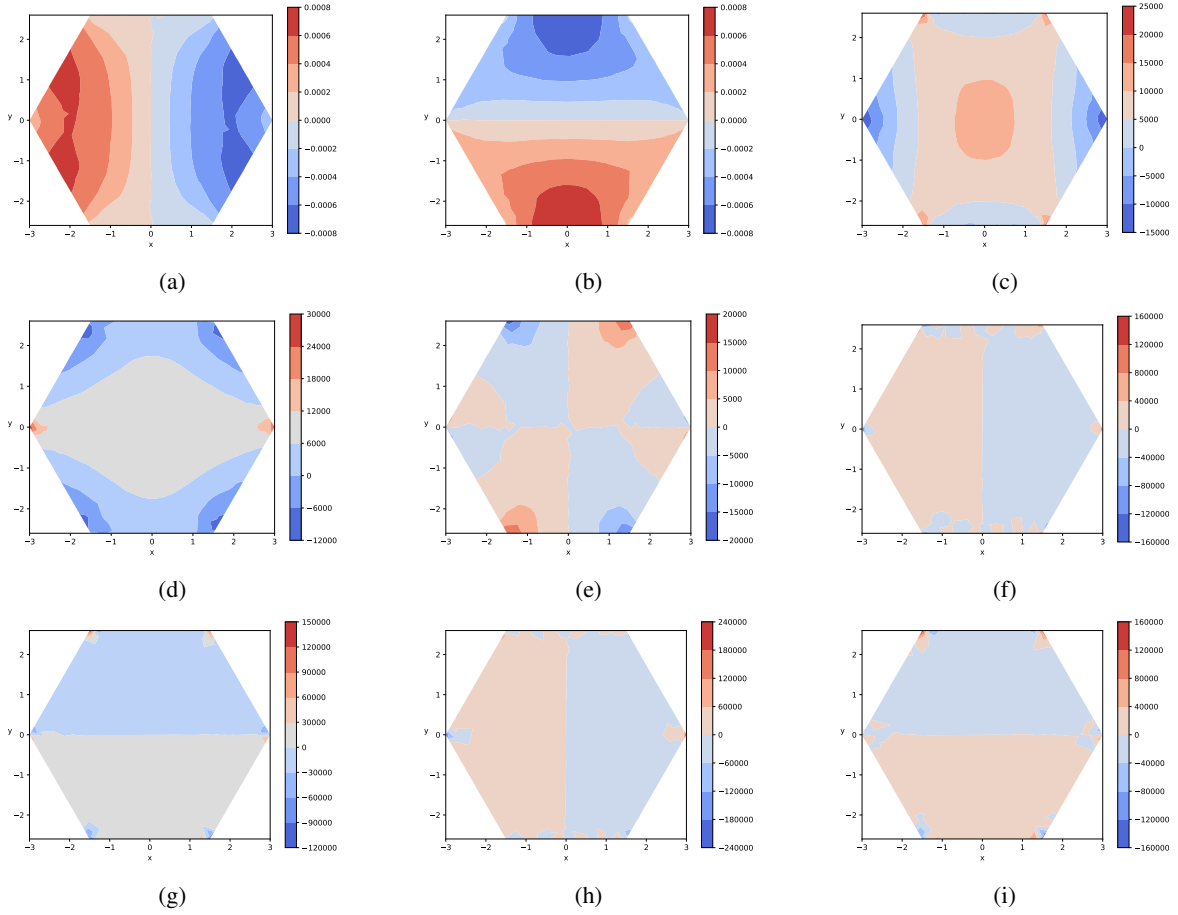


Figure 14: Functions for slopes (a) ϕ_1 , (b) ϕ_2 , bending and twisting moments (c) M_{11} , (d) M_{22} , (e) M_{12} , shearing forces (f) Q_1 , (g) Q_2 and generalized shearing forces (h) V_1 , (i) V_2 of a hexagonal plate

Plots 13 show the deflection of a hexagonal plate for random distribution of surface nodes. The plots of the remaining quantities are presented in Figure 14.

DISCUSSION

The developed method allows solving the problem of bending thin isotropic polygonal plates. The boundary conditions are met with high accuracy at separate edge nodes, but special problems arising at corners with acute and obtuse angles. The problem is known [2, 15] and is related to the presence of singularities occurring at the obtuse corners.

We also see the relationship between the order of the deflection derivative that enters into the expression of the desired quantity. The higher the deflection derivative, the greater the peaks around the corners.

CONCLUSION

Analytical solutions work in selected cases and require an individual approach to the problem each time. The presented method is universal. It can be used to solve plate structures with various configurations, only by changing the input parameters. The current implementation is limited to polygonal plates with convex shapes. In principle, the method should also work for polygonal plates with concave shapes (after introducing some modifications to the node distribution). The generalization of the method requires further development.

Directions for further development include:

1. Elimination of singularities in corners,
2. Uniform arrangement of nodes,
3. Concave polygon-shaped plates.

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